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Representing short-term observations of moving objects by a simple visual language

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Abstract

In a variety of dynamical systems, formations of motion patterns occur. Observing colonies of animals, for instance, for the scientist it is not only of interest which kinds of formations these animals show, but also how they altogether move around. In order to analyse motion patterns for the purpose of making predictions, to describe the behaviour of systems, or to index databases of moving objects, methods are required for dealing with them. This becomes increasingly important since a number of technologies have been devised which allow objects precisely to get traced. However, the indeterminacy of spatial information in real world environments also requires techniques to approximate reasoning, for example, in order to compensate for small and unimportant distinctions which are due to noisy measurements. As a consequence, precise as well as coarse motion patterns have to be dealt with.

A set of 16 atomic motion patterns is proposed. On the one hand, a relation algebra is defined on them. On the other hand, these 16 relations form the basis of a visual language using which motion patterns can easily be dealt with in a diagrammatic way. The relations are coarse but crisp and they allow imprecise knowledge about motion patterns to be dealt with, while their diagrammatic realisation also allow precise patterns to get handled. While almost all approaches consider motion patterns along arbitrary time intervals, this paper in particular focuses on short-term motion patterns as we permanently observe them in our everyday life.

The bottom line of the current work, however, is yet more general. While it has been widely argued that it makes sense to use both sentential and diagrammatic representations in order to represent different things in the same system adequately (and hence differently), we argue that it makes even sense to represent the same things differently in order to grasp different aspects of one and the same object of interest from different viewpoints. We demonstrate this by providing both a sentential and a diagrammatic representation for the purpose of grasping different aspects of motion patterns. It shows that both representations complement each other.

Keywords: Qualitative spatial reasoning; Motion pattern; Change of formation; Relation algebra; Diagrammatic representation

1. Introduction

In a variety of dynamical systems formations of motion patterns occur. The need to deal with them arise for several reasons: patterns of animal movements are investigated providing a detailed picture

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of seasonal variability in the scale and patterns of movements [1]; movement patterns are used as an indicator of cognitive function, depression, and social involvement among people with Alzheimer's disease [2]; spatiotemporal databases require moving objects, such as people, animals, vehicles, or even hurricanes, forest fires, and oil spills to be stored, queried, and retrieved [3]. In particular the need to query such databases indicates the importance of means which are simple to handle by human users. It is our aim to provide a formalism which suffices both the need for a simple tool for describing motion patterns and a thorough computational tool to reason about them. To be clear, in this paper by motion patterns we mean changes in the formation of a number of objects. While there is a great body of work about formations (equally configurations), there is still a lack of methods in order to describe changes of formations—especially when making short-term observations of changing formations. For the description of a traffic scenario, for example, it is less useful to describe a static configuration than to describe how a given formation just changes by considering relative directions of moving objects.

Difficulties in verbalising motion patterns for the purpose of communicating spatiotemporal situations, or in order to index spatiotemporal databases, indicate the importance of means which are simple to handle by human users. Any representation of motion patterns requires both spatial and temporal aspects to be considered. Representing spatiotemporal information, in such a way that in spite of all difficulties the interface between man and machine keeps simple, is a specific problem pertaining to the field of human–computer interaction. We refer to it as to the spatiotemporal representation problem of human–computer interfaces. It relates to the question as to what extent a specific knowledge representation influences human–computer interactions.

From the point of view of cognitive psychology, it has been argued that for human beings graphics serve a variety of functions, amongst others, attracting attention, supporting memory, providing models, and facilitating inference and discovery [4]. From the point of view of the knowledge representation community, it has been argued, that the everyday commonsense knowledge about the spatiotemporal behaviour of objects is captured by qualitative reasoning methods [5]. Reconciling these two views amounts to provide a graphical set of motion patterns which simultaneously form the basis of some qualitative representation. By this means, we will solve a sub-problem of the spatiotemporal representation problem of human–computer interfaces, namely that one for motion patterns. For this purpose we focus on knowledge representation issues and leave questions about the appropriateness of the proposed representation for human users to investigations about human–computer interactions. However, those investigations will built upon the representation we will introduce below, and the diagrammatic realisation of the introduced representation gives first insights in how appropriate the representation is from the point of view of human–computer interfaces since the proposed motion pattern relations can be easily memorised, drawn, and graphically combined.

The main body of this paper is organised as follows. In Section 2 approaches to motion patterns are discussed and it is shown how they lack dealing with an interesting subclass of motion patterns. In the following sections, we shall introduce a representation for this class of motion patterns: Section 3 brings in a set of atomic motion patterns, and a relation algebra, described in Section 4, allows those patterns to be dealt with. After that, Section 5 compares this algebra with a diagrammatic representation, showing what advantages both representations have. Section 6 illustrates the calculus by some examples and we conclude in Section 7 by discussing strengths and weaknesses of our method.

2. Methods for analysing motion patterns

Motion patterns can be analysed by putting emphasis on different aspects. Either patterns of single objects are of interest (i.e. trajectories) or patterns among different moving objects (relations between trajectories). Furthermore, these methods can be classified regarding the entities on which they are based: points (representing positions instead of trajectories), regions (representing extended places instead of trajectories), or lines (representing in fact trajectories or directions of movements); the precision with which movements are described differs along different methods (and relates to the entities used); generally, the intended purposes of different applications call for different demands on the representation to be chosen.

One application area concerns indexing techniques for single moving objects. While Tao and Papadias [6] address the problem of the indexing and retrieval of moving regions, Saltenis et al. [7] provide a method for indexing the current and anticipated future (point-like) positions of moving objects. Similar as Tao and Papadias [6] the approach of Galton [8] is also based on regions. Here, the relationship between moving objects and regions is investigated. In other words, this approach relates places which can be occupied by objects topologically to other regions by the RCC calculus [9]. As in the case of Saltenis et al. [7], the approach of Yaman et al. [10] is based on points. In their so called *go theory*, means for reasoning about moving objects are provided, given start and end positions at which objects are during specified time intervals; additionally, minimal and maximal velocities are given. Among others, it is then possible to check whether a moving object is within a given region at a given point in time.

So far, the methods we have mentioned rely on points or regions. However, in several other approaches line-like entities are employed. Lines might represent trajectories of objects and in particular line segments might represent directions of movement. In Gottfried [11] an approach is introduced that comprises a number of 22 positional and 8 orientation relations which together combine to 125 oriented line–line relations. These relations are especially useful for describing relative movements between objects. In Gottfried [12] the approach is employed for roughly detecting collisions. In its generality the relational system which underlies this approach subsumes others, such as Moratz et al. [13] and Schlieder [14]. As soon as our interests are confined to topological relationships between the traces of objects, one could consider the approach of Egenhofer [15], which is based upon algebraic topology and which compares the interiors, boundaries, and exteriors of lines. By this technique, they identify a total of 33 different topological relations between two simple lines. A combinations of line-like entities which represent the trajectories of objects and regions which represent the object's environment is employed in Gottfried and Witte [16]. Here, the interpretation of movements with regard to their environment (their spatial context) is investigated.

While Yaman et al. [10], Tao and Papadias [6], Saltenis et al. [7], and Galton [8] describe movements of single objects, Nedas et al. [17], Gottfried [11,12], Moratz et al. [13], Schlieder [14], and Egenhofer [15] allow relations among different moving objects to be described. Both movements of single objects and relations between more objects are considered in Gottfried and Witte [16], Yaman et al. [10]. Trajectories of single objects are set into relation to their environment which is divided into regions in accordance to specific application areas [16]. However, this approach also allows trajectories of more than one object to be considered. The approach of Yaman et al. [10] primarily describes the positions of single objects and how they change from location to location; but this approach also enables derivations about more objects. Among others, this approach is able to answer the question whether two objects are within a given distance of each other at some point in time. Common to these and other theories about spatiotemporal representations is that they consider spatial changes in relation to arbitrary temporal intervals. However, a specific subclass of spatiotemporal events relates to short-term considerations, focusing on events which last between, lets say, half a second and half a minute in quite diverse situations: in order to react appropriately in a traffic scenario or in order to let a robot roughly plan what to do next so that it avoids to bump into an obstacle or to let the scientist evaluate the spatiotemporal behaviour of a group of monkeys while some interloper occurs or in order to let the judge analyse a number of testimonies (while A ran towards B, who ran away from A, C ran into the opposite direction of A), etc. Short-term observations are omnipresent. They concern relative movements of a number of objects and they characterise some motion event with a number of objects involved.

Being interested in formations of moving objects as they can be observed by humans, we need methods which rely on distinctions that relate to those distinctions possible by vision. Furthermore, local changes among objects are of interest, i.e. relative changes. This is of interest for the scientist who roughly wants to characterise a scene with moving objects, or who wants to index a database with moving object formations. Such relative changes are considered by Nedas et al. [17], Gottfried [11,12], Moratz et al. [13], Schlieder [14], and Egenhofer [15]. However, Nedas et al. [17], and Egenhofer [15] consider only the intersection of trajectories and provide no means for distinguishing different directions, while Gottfried [11], and Moratz et al. [13] allow for quite many distinctions, more than what is reliably observable by humans. Schlieder [14] eventually allows for both the consideration of directions and a small set of reliably observable distinctions. For our purposes the distinctions possible by Schlieder are then again to coarse since he only distinguishes whether an object moves left of or right of some reference line (although Schlieder [18] shows that there is indeed an interesting class of applications in the context of navigation and wayfinding for which the left–right dichotomy provides already a sufficient distinction). From what follows, we are in need of a representation that allows for only a little more distinctions than what is provided by Schlieder [14]. Therefore, we shall propose a formalism which distinguishes left and right, too, but also forward and backward; and this altogether for the specific case of bipartite motion patterns which form the basis for the characterisation of short-term observations of a number of moving objects.

3. Basic motion patterns

Looking for a small graphical set of motion patterns, we shall analyse what kinds of atomic patterns exist. For such atomic motion patterns we stipulate that they can be obtained by simple observations and that they can be drawn as a simple query-sketch. That is to say, the representation we are looking for makes clear distinctions which do not require any sophisticated measurement tools. Additionally, motion patterns are to be described in a relative way between objects in order to avoid having to consider absolute positions. By this means we shall obtain a formalism that represents relative movement directions of pairs of objects at quite a simple level of description.

3.1. Atomic motion patterns

At least the relative locomotion between two objects are to be described between two time points, t_0 and t_1. This requires the consideration of four points (for both objects at both time points) which are arranged in a specific way depending on how the objects move. These points are to be distinguished in order to represent which point stands for which object at which time point (t_0 or t_1). The most simple graphical way to do this is to connect the four points by three line segments in order to impose an ordering on the points. For example the two middle points could stand for the positions of both objects at t_0 (their start positions) while the outer points show their positions at t_1. From what follows a polyline with three segments represents the relative locomotion among two objects. Depending on the precision required a granularity level has to be chosen for distinguishing such tripartite polylines.

In the most simple case we are concerned with two objects, O and P, each of which moves either towards the other one or away from it, if not straight away then either to the left direction or to the right one. An obvious way of how to represent such distinctions is the orientation grid which has been introduced by Freksa [19], Freksa and Zimmermann [20], and Zimmerman and Freksa [21]. This reference system is induced through two objects and allows their relative positions to be described with regard to both a left-right dichotomy and a towards–away dichotomy. Fig. 1 shows how this reference system is defined: an oriented reference line (depicted as an arrow) connects the positions of O and P; two further lines are defined by the objects and they run perpendicular to the first line. This is an orthogonal, self-referring reference system of two objects using which it is not necessary to employ additionally any external frame of reference. As a consequence, this reference system makes possible a description that is invariant with respect to position, orientation, and scale. Only the objects' identities are to be distinguished as well as which of them serves as the first object that defines what is meant by *left* and *right*. But this is only a matter of definition. In order to simplify the diagrams which we will use below we shall frequently omit the orientation of the medial line (which connects O and P) and define it to be always oriented from left to right with respect to the image plane (the orientation of the medial line is upwards if O and P are above each other).

Then, there are the following possibilities:

- P moves towards O,
- P moves away from O,
- P moves left with respect to O, or
- P moves right with respect to O.

These four possibilities derive from the four obvious directions an object can take. But then, also four

Fig. 1. Left: Two objects O and P define an arrow which connects them. Right: This arrow defines a two-dimensional reference system.

according borderline cases exist between those directions resulting in a total of eight directions, as shown on the left-hand side of Fig. 2. Arguing that it is not necessary to take into account precise directions such as (*straight*) *front*, but rather only a coarse *front* direction, one could choose a reference system like that one on the right-hand side of Fig. 2. Adding the borderline cases to neighbouring directions (as indicated by the brackets), this results in a number of only four directions. If one is, however, still disposed to consider directions easily comprehensible by humans, such as those defined by the two dichotomies of *left–right* and *front–back*, a reference system distinguishing them would be more appropriate than *some* sectors, such as *front* as shown on the right-hand side of Fig. 2; for such a sector it is difficult to say where it ends, in other words, where *left* and *right* start, namely *somewhere* on the *left* and *right*, respectively.

Distinguishing two objects O and P, the possible directions are then shown for O on the left-hand side of Fig. 3 and for P on the right-hand side. Varying these four relations simultaneously for both O and P and combining them we obtain $4^2 = 16$ relations which are depicted in Fig. 4. Note that here the medial lines are to be conceived of to point from left to right. Each relation represents a bipartite motion pattern between two time points, t_0 and t_1, and shows the way two objects take relative to each other during this time interval. While the two endpoints of the middle line define the initial positions of O and P at t_0 (as shown on

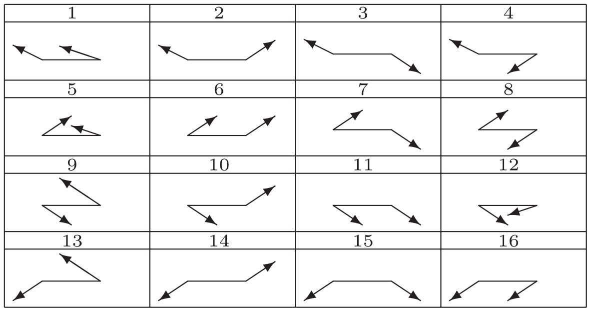

Fig. 4. Sixteen classes of atomic motion patterns.

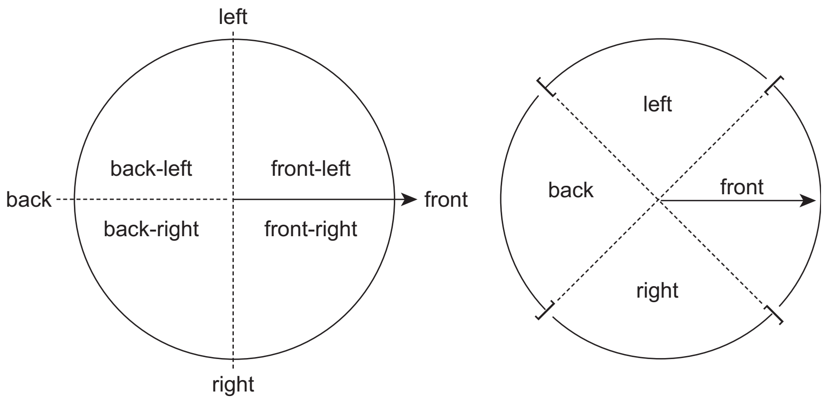

Fig. 2. Two reference systems with eight (left) and four directions (right).

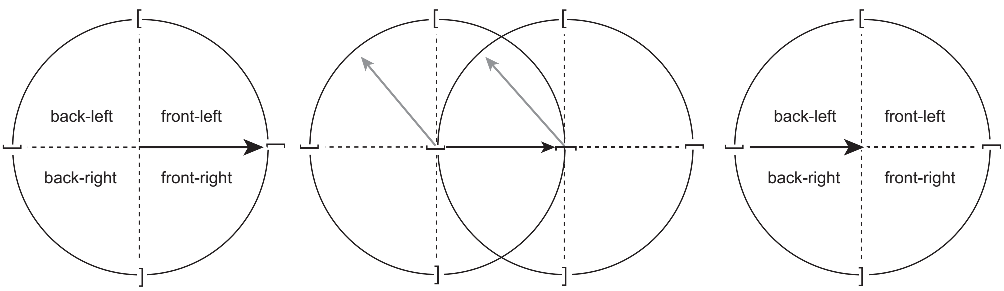

Fig. 3. The quadripartite reference system is induced at the tail of the arrow (left) and at its head (right); the brackets around the circles indicate that precisely front pertains to front–right, precisely right to back–right, and so on. The overlay of both reference systems define the relations (middle), as they are all shown in Fig. 4

the left-hand side of Fig. 1), the heads of the arrows (in Fig. 4) show their target positions at t_1. For the time being, we shall confine the discussion to such bipartite motion formations and shall analyse below how these relations can be applied for describing scenes with more than two objects.

The question as to what each of those 16 relations precisely represents can be answered differently. Each relation can stand for either exact movements or approximations of movements. In the latter case it cannot be excluded that the objects follow quite a complex trajectory during t_0 and t_1. The larger the difference between these two time points, the higher the probability that the objects left the depicted direction for a while. However, their eventual position is shown at t_1, and it depends on what the representation is employed for how to choose an appropriate time interval. In any case, the relations depict what the overall relative direction of movement is for two objects during some time interval.

In Zimmermann and Freksa [21] the orientation grid (which is shown in Fig. 1) is introduced as a means for reasoning about ternary relations. The idea is to describe the positions of objects with respect to the movement vector of somebody walking from some position to another position. This vector induces the orientation grid which in turn is used to describe the positions of the other objects. Therefore, the orientation grid suggests itself to be used for the purpose of describing motion patterns. The relations which describe atomic movements form a subset of those relations which have been defined by Gottfried [22,23]. They used them to characterise shapes and a qualitative concept of curvature information, curvature in fact relating to the concept of direction.

3.2. Linguistic categories

What do those motion patterns tell us? Let us denote the 16 patterns by m_1 to m_{16}. Furthermore, let us agree on O being always the left endpoint of the middle line of the relations in Fig. 4 while P is the right endpoint of that middle line (both at t_0). Then, accordingly to m_1 both O and P move backwards and to the same side with respect to their start configuration; in m_2 they again move to the same side but now only O moves backwards while P moves forward. To give yet another example, in m_6 both O and P move forward and to the same side, namely to the left regarding their start configuration. Specific situations can be derived from those patterns. Those situations are not necessarily mutually distinct and form linguistic concepts which are defined by sets of atomic motion patterns, as those shown in Table 1.

Table 1
Linguistic motion pattern concepts

Concept	Motion patterns
Strong risk of collision	$\{\mathsf{m}_5, \mathsf{m}_{12}\}$
Weak risk of collision	$\{\mathsf{m}_1, \mathsf{m}_6, \mathsf{m}_{11}, \mathsf{m}_{16}\}$
Backwards	$\{\mathsf{m}_1, \mathsf{m}_4, \mathsf{m}_{13}, \mathsf{m}_{16}\}$
Forwards	$\{\mathsf{m}_6, \mathsf{m}_7, \mathsf{m}_{10}, \mathsf{m}_{11}\}$
Opposite	$\{\mathsf{m}_3, \mathsf{m}_8, \mathsf{m}_9, \mathsf{m}_{14}\}$
Depart	$\{\mathsf{m}_2, \mathsf{m}_3, \mathsf{m}_{14}, \mathsf{m}_{15}\}$
Same side	$\{\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_5, \mathsf{m}_6, \mathsf{m}_{11}, \mathsf{m}_{12}, \mathsf{m}_{15}, \mathsf{m}_{16}\}$
Different side	$\{\mathsf{m}_3, \mathsf{m}_4, \mathsf{m}_7, \mathsf{m}_8, \mathsf{m}_9, \mathsf{m}_{10}, \mathsf{m}_{13}, \mathsf{m}_{14}\}$

Table 1 can be conceived of as to be a translation of a natural language description of the change of formations of objects into a diagrammatic and simultaneously formal language which consists of 16 relations. The ambiguities of those concepts, however, do not simply allow a linguistic concept to be translated into one of the 16 relations. But using two or more concepts (opposite and depart) enables one sometimes to get those ambiguities partly to be resolved by intersecting the according sets of motion patterns ($\{\mathsf{m}_3, \mathsf{m}_8, \mathsf{m}_9, \mathsf{m}_{14}\} \cap \{\mathsf{m}_2, \mathsf{m}_3, \mathsf{m}_{14}, \mathsf{m}_{15}\} = \{\mathsf{m}_3, \mathsf{m}_{14}\}$). The more objects there are among which motion pattern constraints exist, the less ambiguous the formal description will be.

3.3. Extensions

Note that we confine ourselves to those situations in which both O and P are in general position at t_1. Including those situations in which objects can be observed precisely, so that it is possible to determine that they move exactly to the left or to the right, we obtain 36 relations. Including additionally those situations in which they can be observed to move precisely forward or backward we obtain 64 relations. Since such observations require precise measurement tools and since they are beyond what people can reliably observe it makes sense to deal with them as follows.

We assign precise relations to other general relations, namely in the way that precisely forward equals forward to the right, precisely backward equals backward to the left, precisely left equals forward to the left, and precisely right equals backward to the right (this is shown in Fig. 3).

However, if such precise distinctions matter they can be included in our formalism without changing any of our assumptions. In any case, we are concerned with a set of jointly exhaustive and pairwise disjoint relations, which is a necessary prerequisite for defining a relation algebra as we will do below.

Similarly, we consider only objects which move and omit stationary objects. But stationary objects can be included in our formalism too, extending only the set of atomic relations. Eventually, we include the identity relation, Id, which describes the movement pattern between two objects, such that they start at the same position and take exactly the same way. Graphically, this amounts to consider a single arrow.

3.4. Neighbourhood based reasoning

Arranging the 16 relations in a neighbourhood graph their similarity can be described by the conceptual distance among them. Fig. 5 shows this graph in which two relations (i.e. vertices in the graph) are connected if they can be transformed into each other by continuously deforming or moving the patterns without passing through another relation, in accordance to Freksa [24]. For instance, m_3 can directly be transformed into m_2, while m_8 cannot directly be transformed into m_2 without passing other patterns during a continuous transformation step.

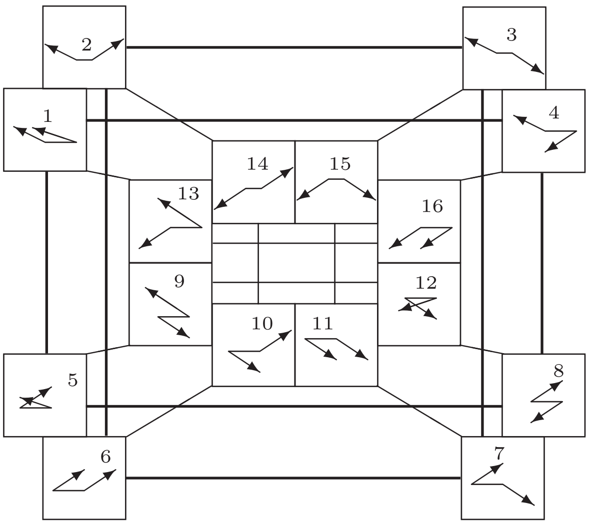

Fig. 5. The neighbourhood graph of the 16 atomic motion patterns.

Note that, in this paper, we confine ourselves to formalise the new set of motion pattern relations. This is for the sake of simplicity and clarity. In any application, however, the relations will be embedded in a more comprehensive logical theory in which also spatial and temporal properties of the objects under consideration and probably also of the environment in which the objects operate are to be defined.

4. A relation algebra on motion patterns

The formal tools which allow relations to be dealt with are relation algebras. Regarding [25] a relation algebra is a nine-tuple:

$$\mathfrak{A} = (\mathscr{M}, \cup, \cap, \bar{}, \emptyset, \mathsf{U}, \circ, \breve{}, \mathsf{Id}), \tag{1}$$

where $(\mathscr{M}, \cup, \cap, \bar{}, \emptyset, \mathsf{U})$ is a Boolean algebra; $\mathscr{M}$ is the universe containing the 16 atomic motion patterns, $\cup$ the union, $\cap$ the intersection, $\bar{}$ the complement, $\emptyset$ is the empty relation, and U the universal relation; $\circ$ is a binary operation called the composition, $\breve{}$ is a unary operation called the converse, and Id is the identity relation. Each relation $\mathsf{m_i}$ contained in $\mathscr{M}$ is an atomic binary relation between two objects, O and P. As the operations on these relations coincide with the usual set-theoretic operations, and since the universe is a set of binary relations, we obtain a proper relation algebra. This is shown in Appendix A.

In comparison to the 16 atomic relations (equally motion patterns), relations $\mathsf{x, y, z}$ are generally defined as sets over the power set of atomic relations: $\mathsf{x, y, z} \in \mathscr{P}(\mathscr{M})$. For any two, not necessarily atomic relations x, $\mathsf{y} \in \mathscr{P}(\mathscr{M})$, $\mathsf{x} \cap \mathsf{y}$ is the intersection of x and y, $\mathsf{x} \cup \mathsf{y}$ is the union of x and y, $\mathsf{x} \circ \mathsf{y}$ is the relative product of x and y, $\breve{\mathsf{x}}$ is the converse of x, and $\bar{\mathsf{x}}$ is the complement of x. For objects O, P, and Q, these operations are defined as follows:

$$\bar{\mathsf{x}} = \{(\mathsf{O}, \mathsf{P}) | (\mathsf{O}, \mathsf{P}) \notin \mathsf{x}\}, \tag{2}$$

$$\breve{\mathsf{x}} = \{(\mathsf{O}, \mathsf{P}) | (\mathsf{P}, \mathsf{O}) \in \mathsf{x}\}, \tag{3}$$

$$\mathsf{x} \circ \mathsf{y} = \{(\mathsf{O}, \mathsf{Q}) | \exists \mathsf{P} : (\mathsf{O}, \mathsf{P}) \in \mathsf{x} \wedge (\mathsf{P}, \mathsf{Q}) \in \mathsf{y}\}, \tag{4}$$

$$\mathsf{x} \cap \mathsf{y} = \{(\mathsf{O}, \mathsf{P}) | (\mathsf{O}, \mathsf{P}) \in \mathsf{x} \wedge (\mathsf{O}, \mathsf{P}) \in \mathsf{y}\}, \tag{5}$$

$$\mathsf{x} \cup \mathsf{y} = \{(\mathsf{O}, \mathsf{P}) | (\mathsf{O}, \mathsf{P}) \in \mathsf{x} \vee (\mathsf{O}, \mathsf{P}) \in \mathsf{y}\}. \tag{6}$$

In expressions without parentheses, the unary operations, i.e. converse and complement, are to be computed first, followed by composition, intersection, and union, in that order; repeated binary operations at the same priority level are to be computed from left to right. The converse operation and the relative product are computed based on the semantics of the relations, which are defined in the table of converse relations Fig. 6 and the composition table (Figs. 7–9), respectively. Note that the identity relation behaves neutrally with respect to composition, that all compositions with the empty relation are empty, and that all compositions with the universal relation result in the universal relation, except for the composition with the empty relation. Thus, it is not necessary to list them in the composition table. Additionally, the table has only entries for atomic motion patterns; for compound relations it is necessary to consider the unions of the compositions of the corresponding atomic relations. All entries found in the composition table are listed in Fig. A1.

It is sufficient to explicitly define only the first four compositions, namely r_1 to r_4 (i.e. $r_1 = m_1 \circ m_1$, $r_2 = m_2 \circ m_1$, $r_3 = m_3 \circ m_1$, and $r_4 = m_4 \circ m_1$). All other relations, and hence their compositions, are symmetrical to the first relations, and as a consequence, to the first four composition

m_1	m_2	m_3	m_4
m_5	m_6	m_7	m_8
m_9	m_{10}	m_{11}	m_{12}
m_{13}	m_{14}	m_{15}	m_{16}

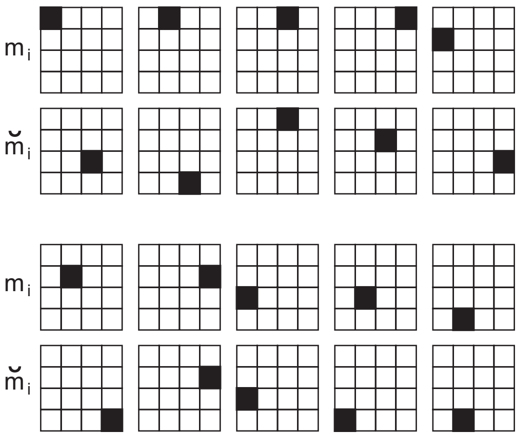

Fig. 6. Left: iconic depiction of the relations according to the arrangement of relations in Fig. 4; right: table of converse relations (note that $\breve{\breve{m}}_i = m_i$); each relation that holds is printed in black; each converse relation is unambiguous.

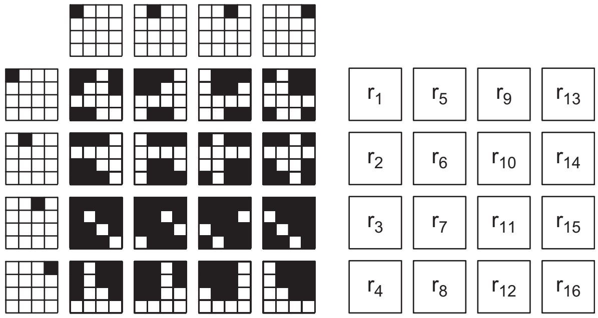

Fig. 7. Non-atomic core compositions r_1 to r_{16}. It holds that $r_5 = r_1^{\rightarrow}$, $r_9 = r_5^{\rightarrow}$, $r_{13} = r_9^{\rightarrow}$, $r_1 = r_{13}^{\rightarrow}$, $r_6 = r_2^{\rightarrow}$, etc. Consequently, only r_1 to r_4 have to been explicitly defined.

r_i → $r_i^{\rightarrow}$

m_1	m_2	m_3	m_4
m_5	m_6	m_7	m_8
m_9	m_{10}	m_{11}	m_{12}
m_{13}	m_{14}	m_{15}	m_{16}

m_4	m_1	m_2	m_3
m_8	m_5	m_6	m_7
m_{12}	m_9	m_{10}	m_{11}
m_{16}	m_{13}	m_{14}	m_{15}

r_i → $r_i^{\downarrow}$

m_1	m_2	m_3	m_4
m_5	m_6	m_7	m_8
m_9	m_{10}	m_{11}	m_{12}
m_{13}	m_{14}	m_{15}	m_{16}

m_{13}	m_{14}	m_{15}	m_{16}
m_1	m_2	m_3	m_4
m_5	m_6	m_7	m_8
m_9	m_{10}	m_{11}	m_{12}

r_i → $r_i^{\circlearrowleft}$

m_1	m_2	m_3	m_4
m_5	m_6	m_7	m_8
m_9	m_{10}	m_{11}	m_{12}
m_{13}	m_{14}	m_{15}	m_{16}

m_{16}	m_{15}	m_{14}	m_{13}
m_{12}	m_{11}	m_{10}	m_9
m_8	m_7	m_6	m_5
m_4	m_3	m_2	m_1

Fig. 8. Operations on the icons, which are defined in Fig. 6. They show the symmetries of the composition table.

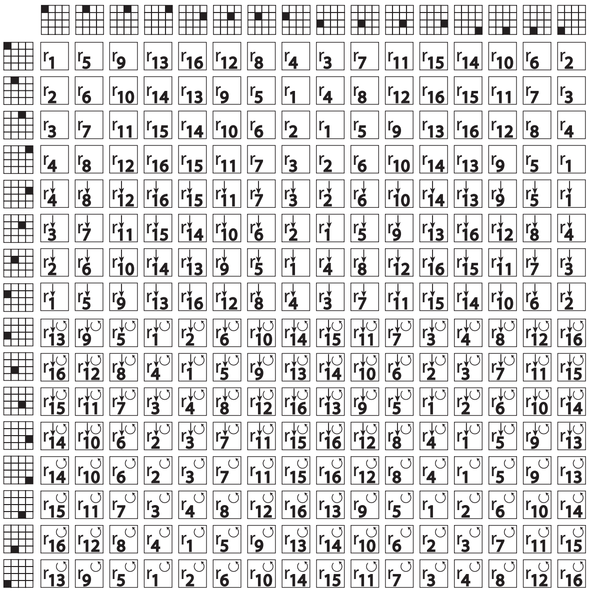

Fig. 9. Composition table.

results. All symmetric cases are obtainable by three matrix operations, f_i, which are defined in Fig. 8. They map non-atomic relations to other non-atomic relations: $f_i : \mathscr{M} \rightarrow \mathscr{M}, f_i \in \{\rightarrow, \downarrow, \circlearrowleft\}$. These matrix operations are only used in order to make explicit the available symmetries. Fig. A1 in the Appendix shows that all occurring non-atomic relations can simply be conceived of as to be specific sets of atomic relations, which can directly be read off Fig. A1.

Making explicit the relative product between two atomic motion patterns, the composition table tells us a great deal about combinations of motion patterns:

(a) All compositions of two atomic motion patterns are valid, and as for the set-theoretic operations the relations are closed under weak composition (i.e. the composition operation results in sets of relations instead of single relations).

(b) There is no unique inference result (except for those compositions with the identity relation). Even when precise observations can be made, generally only after one inference step knowledge becomes indeterminate. Conversely, there is no inference result which is completely indeterminate, such as for the composition result of two *before* relations of Allen's calculus [26]. Instead, there are only inference results with either 7, 8, or 13 relations, i.e. restrictions in order to reduce search space can always be made.

(c) All inference results form conceptual neighbourhoods of disjunctions of atomic motion patterns in the sense of Freksa [24]. Note that each composition result is an icon accordingly to Fig. 4, but that the neighbourhood relations hold accordingly to the neighbourhood graph in Fig. 5.

(d) From the $2^{16} = 65\,536$ possible elements of the powerset $\mathscr{P}(\mathscr{M})$ only 64 exist as composition results of atomic relations, plus those 16 unique composition results with the identity relation, plus the empty relation, resulting into a total of $64 + 16 + 1 = 81$ different possible composition results.

Having introduced a relation algebra on $\mathscr{M}$, we are able to apply constraint based reasoning techniques on the relations of $\mathscr{M}$. Above we described relations in the set-theoretic way, but express constraints between two objects O and P also by writing $\mathsf{O}\ m_i\ \mathsf{P}$; or we write $\mathsf{O}\{m_i, m_j, \ldots\}\mathsf{P}$ if there are two or more relations which are assumed to hold between O and P.

5. A diagrammatic representation on motion patterns

In the previous section we have learned that the composition results are always ambiguous. This is an unpleasant fact that derives from the coarseness of the relations. In other words, with the relation algebra we are faced with a weak composition definition, however, which is found in other relation algebras too [9,11,13,19,26]. At least, whenever being faced with an observation that allows the start configuration accurately to be described, we would like to have a tool which allows unambiguously compositions to be derived. Moreover, such a tool should be capable of illustrating observations, aiding in analysing and communicating them, helping in spatial planning, or even in establishing existential proofs. Such a (diagrammatic) tool is introduced in Section 5.1. Afterwards, Section 5.2 relates this diagrammatic system to the relation algebra introduced in Section 4. Then, in Section 5.3 further differences between the algebraic and the diagrammatic systems are discussed. Section 5.4 summarises the usefulness of the diagrammatic system.

5.1. The diagrammatic representation

In a diagram it is only possible to depict one specific instance for each relation in $\mathscr{M}$, although there are infinite many instances for each relation possible. This is the advantage of the algebra which has been introduced in the previous section: it enables us to deal with infinite many instances at a sentential level and as a consequence to exhaustively describe all situations. As soon as we are able to accurately describe some situation, however, we are not interested in exhaustively describing all possible realisations of the given relations. Instead, we want to represent exactly what we observe and we only want to derive conclusions from what has been observed, which is less ambiguous than what exhaustively is captured by the algebra. In this case, the confines of the diagrammatic realisation is what we are looking for: we are just interested in one particular realisation of the relations given.

As a consequence, whenever a specific observation can be depicted its composition should be derived diagrammatically, instead of algebraically. For this purpose, a simple diagrammatic construction process is to be applied: those pairs of objects for which we want to know their relation (which is algebraically obtained by composition) are to be connected by a straight, directed line. A subsequently applied inspection process simply consists in comparing this constructed relation with the set of 16 atomic relations. Eventually, there will be found precisely one relation which matches the constructed one. This is the diagrammatic, and hence strong, composition result. A diagrammatic system working in this way can be formalised as follows:

A diagrammatic representation, $\mathfrak{D}$, is a six-tuple: $\mathfrak{D} = (\mathbb{R}^n, \mathbf{O}, \mathrm{R}, \mathscr{C}, \mathscr{M}, \mathscr{I})$. It consists of an n-dimensional space, $\mathbb{R}^n$, a set of objects, $\mathbf{O}$, which can be embedded in $\mathbb{R}^n$, and a set of relations R which are defined among the objects in $\mathbf{O}$.

$\mathscr{C}$ is a set of functions each of which allows the construction of a diagram, given a number of objects: $f_{\mathscr{C}} : \mathbf{O} \rightarrow \mathbb{R}^n$;

$\mathscr{M}$ is a set of functions each of which allows the modification of a diagram: $f_{\mathscr{M}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$;

$\mathscr{I}$ is a set of functions each of which allows the inspection of a diagram: $f_{\mathscr{I}} : \mathbb{R}^n \rightarrow \mathrm{R}(\mathbf{O})$, each inspection function resulting in a number of relations $R \subseteq \mathscr{P}(\mathrm{R})$ among the objects in $\mathbf{O}$.

Many diagrams, and especially those in which we are interested in, are two-dimensional, i.e. $n = 2$. Then, $\mathbf{O}$ might contain every kind of objects which can be embedded in the plane (points, line segments, polylines, etc.). Here, point-like positions are of interest, and therefore, $\mathbf{O}$ is a set of points. Moreover, $\mathbf{O}$ contains straight line segments and straight arrows, used to connect the positions of objects.

Construction functions map positions to the two-dimensional plane. Here the start and end positions of pairs of objects are of interest which is why there are four points that map onto the diagram in order to describe a basic motion pattern. These four points are connected in accordance to the relations and their semantics defined in Section 3. In this way, construction functions are to provide well defined diagrams. A number of basic motion patterns can be diagrammatised simultaneously. The left-hand side of Fig. 10 shows an example with two motion patterns (both m_7) among three objects.

Then, two kinds of modification functions are distinguished. Firstly, those which construct oriented, straight line segments among specific pairs of points, allowing to determine relative movements (this operation is denoted by $\circ_{\mathfrak{D}}$, that is $m_i \circ_{\mathfrak{D}} m_j$ introduces a new straight line segment between p_k and p_l which denote the start positions of two of the objects involved in the two motion patterns m_i and m_j). The middle of Fig. 10 introduces another straight line segment between two objects, making explicit the motion pattern holding among these objects (which is m_3).

Secondly, there are those modifications which change the orientation of specific oriented line segments among two points, allowing to determine the converse of a given atomic motion pattern (we denote this operation by $\breve{m}_{\mathfrak{D}}$ for motion pattern m). On the right-hand side of Fig. 10 the direction of one of the straight line segments between two objects is changed, making explicit the converse motion pattern holding among these objects (which is m_4).

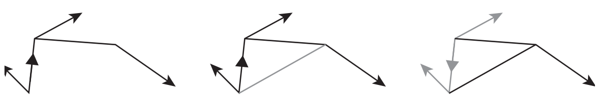

Fig. 10. The construction of a diagram with three objects (left), the introduction of a modification making explicit another motion relation (middle), and a modification showing the converse of a relation m_7, which is m_4 (right).

Inspection functions determine relations between the objects in **O**, in the present case relative movements that are characterised by the 16 atomic motion patterns contained in $\mathscr{M}$ (see Section 4).

5.2. Distinctions between algebra and diagram

How does this diagrammatic representation relate to the algebraic representation? We recognise that the relations and operations of the relation algebra, $\mathfrak{A}$, map to diagrammatic relations and operations of $\mathfrak{D}$. This mapping is uniquely defined for all operations except for the composition operation which is unique in $\mathfrak{D}$ but not in $\mathfrak{A}$ (compare Fig. 11), so that each unique composition result in $\mathfrak{D}$ (obtained by the diagrammatic composition $\circ_{\mathfrak{D}}$) is an element of the ambiguous composition result that is obtained algebraically (and which is denoted by $\circ_{\mathfrak{A}}$). That is, for any pair of motion patterns their diagrammatic composition is an element of the ambiguous composition result of $\mathfrak{A}$, as can be seen in the composition table in Fig. 9:

$$\forall_{m_i,m_j \in \mathscr{M}} : m_i \circ_{\mathfrak{D}} m_j \in m_i \circ_{\mathfrak{A}} m_j \subseteq \mathscr{M}. \tag{7}$$

There are, however, two exceptions since the identity relation behaves neutrally with respect to composition and since compositions with the empty relation are empty, so that in these cases the algebraic compositions are uniquely defined too. However, since the relations of $\mathfrak{D}$ map surjectively to those of $\mathfrak{A}$ we are not concerned with an isomorphism between $\mathfrak{A}$ and $\mathfrak{D}$.

A second distinction between $\mathfrak{A}$ and $\mathfrak{D}$ is that $\mathfrak{A}$ captures the whole set of possible instances for each relation of $\mathscr{M}$ to which $\mathfrak{A}$ refers to, while in $\mathfrak{D}$ each relation can only be depicted by one specific instance. In fact, this is the reason for the diagrammatic composition, $\circ_{\mathfrak{D}}$, to be unique (only the composition for one specific instance is diagrammatically computed) and the algebraic composition, $\circ_{\mathfrak{A}}$, being ambiguous. To summarise, the difference between $\mathfrak{A}$ and $\mathfrak{D}$ consists in the precision with which $\mathfrak{D}$ refers to specific relations, and as a consequence, with which $\mathfrak{D}$ allows a precise composition result to be derived (Fig. 11, right hand side).

Besides the atomic relations and the composition operation all other components of $\mathfrak{A}$ map uniquely to components of $\mathfrak{D}$: the identity relation maps to a single arrow in $\mathfrak{D}$; the converse relation is also unique in $\mathfrak{D}$, the arrow of the start configuration is only to be changed towards the other direction which amounts to a change of 180° (cf. Fig. 12); the universal relation includes every conceivable diagrammatic relation; the empty relation is that one in $\mathfrak{D}$ which is impossible to realise in the two-dimensional plane; all other set-theoretic operations of $\mathfrak{D}$ correspond to those in $\mathfrak{A}$, again in each case with the infinite range of possibilities in $\mathfrak{D}$.

5.3. It depends on the problem…

…whether using $\mathfrak{A}$ or $\mathfrak{D}$. Being faced with a precise problem description we prefer to employ $\mathfrak{D}$ in order to obtain precise composition results. By

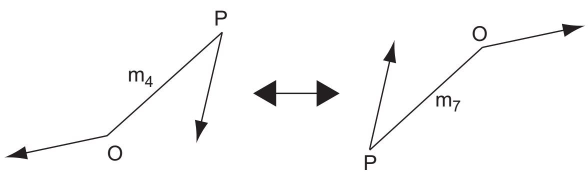

Fig. 12. The converse of m_4 is m_7 and vice versa. In $\mathfrak{D}$ only the arrowhead is to change in the start configuration of O and P (or since we left out the arrow head of the middle line and defined each line segment to be oriented from left to right regarding the image plane, the converse amounts to a change in direction by 180°).

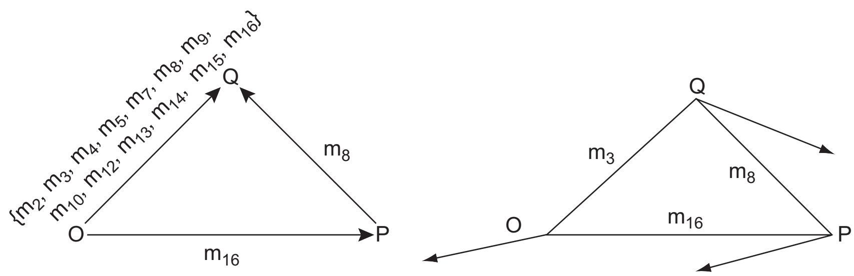

Fig. 11. A constraint graph and the ambiguous algebraic composition result of $m_{16} \circ m_8$ (left) and the diagrammatic solution for a precise instance (right).

contrast, having only a coarse problem description we have to employ $\mathfrak{A}$, which is still better than to have no tool at all for reducing the search space of possible relations. In the rest of this section we shall analyse further differences among $\mathfrak{A}$ and $\mathfrak{D}$, allowing us to make decisions on which representation to use for a given problem.

5.3.1. Change of relation

Changing relations entails different consequences for both representations:

$\mathfrak{A}$: delete the old relation and enforce the new one: compute all consequences by the converse operator and probably by composition as far as no direct relation exists for two given objects—this has to be done for all pairs of objects regardless of whether this is of interest for all pairs of objects or whether not.

$\mathfrak{D}$: delete the old edge and draw the new one; read off all consequences by inspecting the changed diagram—this can be restricted to those relations which involve the new edge because only they show new relationships.

5.3.2. Change of velocity

While $\mathfrak{A}$ ignores any target positions to be specified, $\mathfrak{D}$ allows for velocities and thus for target positions to be considered (directions are already given by the relations themselves). If all motion arrows are equal in length, all objects move equally fast. One could either ignore velocities (as $\mathfrak{A}$ does), one could distinguish whether one motion arrow is shorter than another one (which amounts to include a *faster-than* relation in $\mathfrak{D}$), or one could even take into account precise velocities in $\mathfrak{D}$. Then, changing velocities simply means to shorten or elongating the motion arrows. This shows how $\mathfrak{D}$ allows more information (than directional information) easily to be integrated.

5.3.3. Combination with other information

While for $\mathfrak{A}$ the combination with further information, for instance, topological information, showing in which kinds of regions movements take place, has to be defined explicitly, this is much simpler for $\mathfrak{D}$. In the diagrammatic case just the overlay of properly sized and properly aligned diagrams are to be considered. Gottfried and Witte [16] provide a formalism for dealing with spatially contextualised motion patterns. Here again, it shows how $\mathfrak{D}$ allows further information more easily to be integrated than $\mathfrak{A}$.

5.4. Applying diagrams

There are primarily the following classes of applications, which show that there are a number of cases for which the diagrammatic representation makes sense: (a) observations are made and (b) visualised, (c) queries are to be specified, (d) hypotheses are to be tested, or (e) plans are to be elaborated or to be changed:

(a) Depending on how precise observations are it makes sense either to use $\mathfrak{A}$ (imprecise observations) or to use $\mathfrak{D}$ (precise observations).
(b) Visualisations are useful to communicate situations or to analyse them.
(c) For the purpose of specifying queries in order to look for motion patterns, a diagram is made and translated into (general) algebraic relations, which are used in order to index a database.
(d) In order to corroborate a hypothesis (which amounts to an existential proof) it is only necessary to show that there exists at least one instance, hence this can be done diagrammatically.
(e) For the purpose of planning, one should not abandon the option to make a sketch. It consists in creating graphically relations and in showing only for a specific instance that this will work in accordance to the intended purposes. Changing this diagram allows changes of the plan (and the consequences of these changes) easily to get comprehended.

A final example illustrates the capabilities of $\mathfrak{D}$ as a tool for visualisation and for supporting our intuition in diagrammatic proofs. One is interested in knowing whether formations with an arbitrary number of objects exist, so that each pair of objects departs. We refer to this problem as to the *n-disperse problem* which is inverse to what has been introduced before as the *convergence pattern* [27]. While there is a broad community of mathematicians who reject any diagrams as parts of proofs, they have been proposed by others, such as Jamnik et al. [28] who discuss theorems of mathematics that admit diagrammatic proofs. For instance, the sum of odd naturals and the geometric sum of $\frac{1}{n^2}$ have been diagrammatically proven.

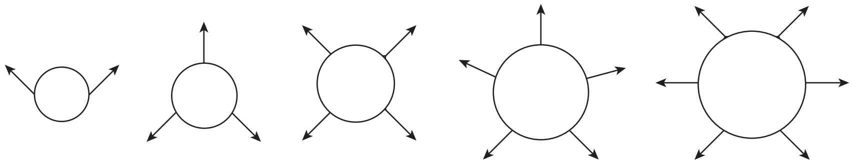

Fig. 13. Placing n (= 2, 3, 4, 5, 6) objects on a circle at t_0, all running into different directions.

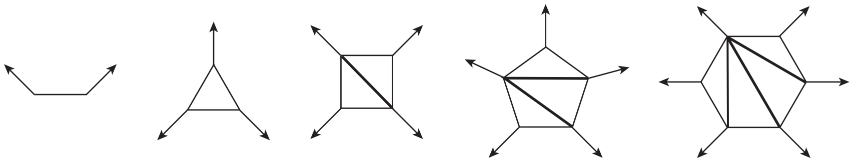

Fig. 14. *Disperse relations* among all pairs of objects, within formations of n (= 2, 3, 4, 5, 6) objects. Some of the symmetrical segments are omitted for clarity.

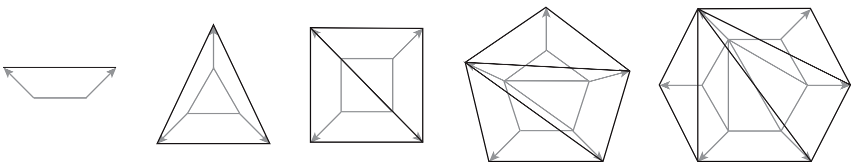

Fig. 15. All distances (black) among the target positions have become larger.

Here, we do not claim that our diagrams can be conceived of as something similar to formal proofs, but that they support human reasoning in a way similar like illustrations in a textbook; although, they go one step further by not only providing single instances of some proposition but by providing a general digrammatic strategy how to construct or manipulate motion pattern formations, for example, by extending them step-by-step in order to show the possibilities which are factored into a finite number of cases, however, in a way that one is disposed to believe that the proposition will hold in every case, since only specific relations of $\mathscr{M}$ occur. In this sense, Figs. 13 and 14 show a diagrammatic proof of why the *n-disperse problem* possesses a solution; it visualises the existence of solutions (produced by an appropriate construction function which arranges objects on the boundary of a circle, as shown in Fig. 13). Then, as Fig. 14 shows in such formations one of the following relations holds between every pair of objects: m_2, m_3, m_{14}, or m_{15}, which is the result of both a modification function that introduces line segments between start positions at t_0 (Fig. 14) and an inspection function that reads off all relations. The diagrams support our intuition that this can be done for every $n \geqslant 2$.

Arranging objects on a circle it is quite obvious (without $\mathfrak{D}$) that the objects disperse, however, with the relations in $\mathscr{M}$, $\mathfrak{D}$ gives an explanation of why this is so.

Fig. 15 additionally shows that the distances between all pairs of objects get larger after they have dispersed (by a simple digrammatic modification function that introduces (black) line segments, connecting the target positions at t_1; an inspection function tells us afterwards that the introduced line segments are larger than the segments between the positions at t_0). Fig. 16 eventually shows the motion patterns necessary for the objects to meet again (i.e. to converge). Properties of other formations and their change can be analysed accordingly with the diagrammatic means of $\mathfrak{D}$, frequently by showing that specific subsets of $\mathscr{M}$ hold (and there exist 2^{16} of them!), when applying specific construction and modification functions to a number of n objects.

6. Example applications

A number of examples illustrate the method introduced in the previous sections. At first, the satisfiability problem for a set of relations is discussed, showing how to apply constraint based reasoning techniques to motion patterns. After that, some examples show how things can be simplified in specific scenarios, and also these

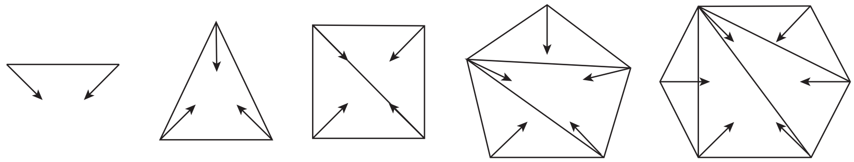

Fig. 16. If the objects again want to come together, two motion patterns and their converse relations are required: m_5, m_8, m_9, and m_{12} define convergence patterns.

	O	P	Q	R
O	Id	U	2	13
P	12	Id	9	U
Q	U	U	Id	U
R	U	4	2	Id

	O	P	Q	R
O	Id	5	2	13
P	12	Id	9	7
Q	15	9	Id	15
R	10	4	2	Id

	O	P	Q	R
O	Id	5	∅	13
P	12	Id	9	∅
Q	∅	9	Id	∅
R	10	∅	∅	Id

Fig. 17. A constraint network: node-consistent, arc-consistent, and path-inconsistent.

examples demonstrate how concrete problems translate into sets of specific motion patterns.

6.1. Consistency of knowledge

An important problem is the satisfiability problem. In our case we refer to it as MPSAT, abbreviating *motion pattern satisfiability*. Given a set of motion patterns $\mathbb{M}$, MPSAT asks for the consistency of $\mathbb{M}$, i.e. whether a position can be found for a number of objects, so that all constraints in $\mathbb{M}$ are simultaneously satisfied. MPSAT is a constraint satisfaction problem (CSP) and can be solved using standard methods.

The computational evaluation of the consistency of a constraint net is performed as follows. In order to achieve arc-consistency

$$\forall_{O,P} : Or'P := OrP \cap P\breve{r}O, \quad r, r' \in \mathscr{P}(\mathscr{M}). \tag{8}$$

and in order to achieve path-consistency

$$\forall_{O,P,Q} : Or'P := Or_1P \cap (Or_2Q \circ Qr_3P), \\ r_1, r_2, r_3, r' \in \mathscr{P}(\mathscr{M}). \tag{9}$$

These two steps are to be performed until no new relations are inferred. As the empty relation denotes an inconsistent scenario, as soon as the empty relation is deduced, the constraint net will have been proven to be inconsistent because any empty relation will remain empty under any further computations of Eqs. (8) and (9). When the network has stabilised without inferring the empty relation the constraint net has been shown to be path-consistent.

For instance, we are concerned with four objects, O, P, Q, and R, and a number of six constraints among them: $\mathbb{M} = \{O\ m_2\ Q, O\ m_{13}\ R, P\ m_{12}\ O, P\ m_9\ Q, R\ m_4\ P, R\ m_2\ Q\}$. The tabular representation of this constraint network shows the given constraints. The relations between the other objects are not constrained, i.e. we have no information about those relations and the universal relation holds between those objects. Furthermore, the identity relation holds for the relation between each object and itself, and we obtain the constraint network which is shown on the left-hand side of Fig. 17. After having applied Eq. (8), we obtain an arc-consistent network in which each relation is constrained (the middle table of Fig. 17). For the purpose of achieving a path-consistent network, we apply Eq. (9) and obtain the rightmost table in Fig. 17. It turns out that the network is inconsistent since the empty relation could be inferred six times. As such, it is impossible to draw a diagram with four objects among which the constraints of $\mathbb{M}$ hold.

Note that MPSAT comes up only for $\mathfrak{A}$, since every set of motion patterns depicted in a diagram $d \in \mathfrak{D}$ is consistent since inconsistent scenarios cannot be depicted in the plane. As such, an application of MPSAT consists in deciding whether a set $\mathbb{M}$ of given motion patterns can be realised at all on some display in the context of user interfaces. If so, $\mathbb{M}$ is consistent and a well defined diagram of $\mathfrak{D}$ exists.

6.2. Integration of knowledge

The procedure described in the previous section can also be applied in order to integrate new knowledge. Each new relation extends the set $\mathbb{M}$, and we update the constraint network by propagating such further constraints. Similarly, knowledge which is no longer valid can be considered by updating the constraint net after the corresponding relations in $\mathbb{M}$ have been deleted.

But the integration of knowledge can frequently be reduced to the intersection of constraints which have been acquired from different sources. For instance, assuming two cameras monitoring a complex crossroad, one camera observes that two objects, O and P, move towards each other, but it cannot decide whether they will bump into each other, i.e. $O\{m_5, m_8, m_9, m_{12}\}P$. Another camera observes that same scene from a different viewpoint. By contrast to the first camera, that one can only perceive O and P to move towards the same side—either parallel or not, i.e. $O\{m_1, m_2, m_5, m_6, m_{11}$ $m_{12}, m_{15}, m_{16}\}P$. However, both cameras suppose that their observations are correct (albeit coarse). Hence, the intersection of their observations is taken in order to allow a more precise statement to be made, and the conclusion is that $O\{m_5, m_{12}\}P$, i.e. danger is ahead.

For $\mathfrak{D}$ the integration and updating of knowledge is simpler. Objects can be removed and their connections to other objects accordingly; or objects can be inserted and connections which are of interest can be drawn, indicating which relations hold among new objects and among new ones and others. Modification functions of $\mathfrak{D}$ can be restricted to objects which are of interest for a current problem.

6.3. Disproof of knowledge

Finally, we shall consider an example in which a claim is proved to be wrong in a civil hearing. Accordingly to a number of testimonies, O and P ran into opposite directions, away from each other. At the very same time P and Q separated, walking into different but not opposite directions. Eventually, O and R walked towards each other, though on different sides of some object. Is it possible that R and Q is one and the same person, as claimed by a witness?

Formalising these evidences we obtain the following relations:

- $O\ m_3\ P$; though m_{14} would have also be a possible choice. However, one can choose between those relations. But as soon as we commit ourselves to one of those relations, ensuing choices have to be made accordingly.
- $P\ m_{15}\ Q$; here, we have to take m_{15} instead of m_2 since we decided for m_3 before.
- $O\{m_8, m_9\}R$.

If R and Q are equal, the relations between O and Q as well as between O and R have to be equal, too. We obtain the relation between O and Q by composition:
$O \circ Q = \{m_1, m_2, m_3, m_5, m_6, m_7, m_{10}, m_{11}\}$. As it holds that $\{m_8, m_9\} \notin O \circ Q$, we conclude that R and Q cannot be the same person.

$\mathfrak{D}$ can now be employed for illustrating the scenario and for communicating it. Fig. 18 illustrates this. The assumptions are shown in the upper part of the figure, while the lower part shows how these assumptions combine graphically. The right-hand side of the lower part shows that it is neither possible to realise $O\ m_8\ Q$, nor $O\ m_9\ Q$. We conclude diagrammatically as we did before, i.e. that R and Q are different persons.

Other application areas are collaborating agents who also act in space, video analysis methods for which a symbolic description of how objects move

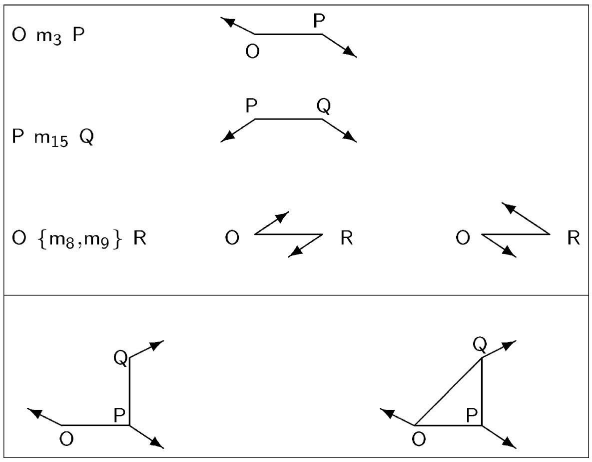

Fig. 18. Upper part: graphical depiction of the observed motion patterns (premises); lower part: the combination of $O\ m_3\ P$ and $P\ m_{15}\ Q$ (left)—resulting in $O\ m_3\ Q$.

are relevant, or systems for natural language understanding which are in need of commonsense representations of moving objects. In all these areas qualitative spatial representations aid in representing the dynamic aspects of a number of objects at a conceptual level.

7. Discussion

Having introduced a qualitative representation of motion patterns, we will point out some of its strengths and weaknesses. This helps in deciding for which problems the representation will be an appropriate tool.

(a) It is sometimes necessary to take into account precise directions and precise velocities when describing motion patterns. On the other hand, whenever precise information is not available, imprecise knowledge about motion patterns is probably better than to have no information at all. However, in this case we need an appropriate formalism in order to deal with such kind of knowledge.

(b) Indeterminate knowledge is dealt with as follows. Incomplete information can be represented by sets of atomic motion patterns. If we lack any knowledge about the relationship between two objects, the universal relation U holds between them. Imprecise information is inherently represented by the atomic relations. Each one represents a great range of variations, though perceptually clear distinctions are represented by different relations.

(c) By contrast to probabilistic methods which require assumptions such as a priori probabilities or knowledge about frequency distributions the described techniques are not in need of such knowledge. They require only observations allowing to state which relations hold. Probabilistic methods are more appropriate if it is necessary to reason about fine deviations and distinctions among motion patterns. If we are only interested in rough distinctions a qualitative representation might be sufficient.

(d) The usage of a number of qualitative distinctions leads to more efficient representation and reasoning techniques than when using precise quantities and variables with continuous domains. This should in particular be seen in line with a decision making system. Decisions are frequently of binary nature and a concise qualitative assessment of the state of affairs already allows to some extent conclusions to be made, such as whether two moving objects might collide or whether they depart.

(e) The motion pattern relations introduced are clearly laid out, show apparent distinctions, and as a consequence, are closely related to language as has been demonstrated in particular in Section 3.2. But they can also be communicated in a graphical way, since those patterns differ only with regard to 90° angles. Therefore, they especially form appropriate means for querying databases in the sense of spatial-query-by-sketch systems (cf. [29]). A sketch analysing system would be quite robust regarding those coarse distinctions and would have had the advantage to avoid linguistic concepts which comprise a lot of ambiguities that would make the indexing of databases with such dynamical knowledge quite complex.

(f) The diagrammatic representation can be extended in several ways. According to Lindsay [30] a number of diagrammatic construction functions enable the graphical depiction of some scenario, and succeeding inspection routines would allow relations to be read off the diagram which had not been explicitly stated before: their overlay with some map, for instance, would enable to read off the diagram new relationships between moving objects and their environment.

(g) As in the case of other qualitative representations this approach suffers from a weak composition definition. That is, inferences made by the composition table result always in sets of possible relations, instead of unique results. After all, 43.75% of the relations possible can be excluded in the average in one inference step, as far as atomic motion patterns are used as premises for the composition operation. Therefore, it is absolutely reasonable to reduce search space by these relations and their relative product. From what follows, the proposed techniques can either be used for trimming search space or they will even be sufficient to completely solve a problem in which constraints mutually restrict the degree of indeterminacy, so that finally relationships become unique.

(h) Besides the application of the algebra the representation also provides means for neighbourhood based reasoning. For example, given some relation it is possible to predict relations that might follow next and it is possible to exclude other relations by the neighbourhood graph. Additionally, this graph can also be used to measure the similarity of motion patterns by taking the distance between relations in this graph.

(i) While a set of 16 qualitative atomic relations still exhibits a manageable size, including more relations (such as precisely left and right, and precisely forward and backward) the set of atomic relations becomes quite large, and as a consequence more difficult to handle in the context of adequate user interfaces. Certainly, exploiting the symmetries inherent in the orientation grid things might still remain manageable (at least from the point of view of implementation). However, it is questionable to include such precise relations into a qualitative framework for yet another reason: It is sometimes just because of the difficulties to obtain precise information that qualitative representations are used at all. Another alternative worth mentioning would be to split the space into 45° sectors, resulting in directions such as north, north-east, etc. Such a representation would be less coarse but it should still be of a manageable size since it comprises only twice as much directions as the representation used. Then, there would be eight borderline cases which could be handled likewise demonstrated in Fig. 3. Similarly, even more distinctions could be made if necessary, requiring however, to introduce the according algebras, in particular with defining their composition operations which would become more precise the finer the distinctions used.

8. Summary

To summarise, we identified a number of 16 atomic motion patterns, which describe the relative motion between two objects regarding both the left–right and the towards–away dichotomy. On the one hand the relations form the basis of a relation algebra that provides a powerful reasoning tool, especially when being faced with imprecise observations. On the other hand their diagrammatic realisation forms a representation which enables one to precisely deal with specific situations. Taking together both representations the same motion patterns are viewed from different perspectives—both perspectives offering different advantages. In this sense, this paper argues in favour of simultaneously representing objects differently for the purpose of capturing different aspects of them, for example, different levels of precision and corresponding tools to deal with them. Altogether, the proposed relations are easily accessible from the point of view of the user and their application makes sense in such diverse fields as in user interfaces, databases, planning, and spatiotemporal reasoning.

Appendix A. Relation algebra

In this Appendix we shall show that the introduced algebra is in fact a relation algebra. For this purpose, we use the following abbreviations:

$\mathcal{M}$	denotes the set of 16 atomic motion patterns (this is the universe of basic relations)
$\mathsf{m_i}$	denotes a single atomic motion pattern $\mathsf{x, y, z}$ denote non-atomic motion patterns (i.e. sets of atomic motion patterns)
$\mathsf{r_i}$	denotes a specific, non-atomic motion pattern (i.e. a specific x, as used in the composition table)
Id	the identity relation
$\emptyset$	the empty relation
U	the universal relation

Moreover, the following operations, which are defined in Section 4, are used:

$\cup$	union
$\cap$	intersection
$\bar{}$	complement
$\circ$	composition
$\breve{}$	converse
$r_i^{\rightarrow}$	denotes a function that transforms r_i into another non-atomic motion pattern according to Fig. 8
$r_i^{\downarrow}$	denotes a function that transforms r_i into another non-atomic motion pattern according to Fig. 8
$r_i^{\circlearrowleft}$	denotes a function that transforms r_i into another non-atomic motion pattern according to Fig. 8

According to Ladkin and Maddux [25] a relation algebra satisfies seven axioms:

$$(x \circ y) \circ z = x \circ (y \circ z) \tag{1}$$
$$(x \cup y) \circ z = x \circ z \cup y \circ z \tag{2}$$
$$x \circ Id = x \tag{3}$$
$$\breve{\breve{x}} = x \tag{4}$$
$$(x \cup y)\breve{} = \breve{x} \cup \breve{y} \tag{5}$$
$$(x \circ y)\breve{} = \breve{y} \circ \breve{x} \tag{6}$$
$$\breve{x} \circ \overline{x \circ y} \cap y = \emptyset \tag{7}$$

Since the second equality can be derived from the others [25], we only show 1 and 3–7. Furthermore, as in the case of the composition definition we make use of the available symmetries, and explicitly prove the axioms for every pair of relations defining the first column in Fig. 7. In this way we capture all cases, namely those with both objects moving towards the same direction (as in m_1), moving towards different directions (as in m_2), moving opposite to each other to different directions (as in m_3), and moving into different directions, though not opposite to each other (as in m_4). Thus, the proofs of the axioms are explicitly to be shown for m_1, m_2, m_3, and m_4.

(1) Since the associativity of the composition operation implies to take the relative product two times the universal relation is always obtained, since each composition operation, as the composition table shows, results at least in a set of seven atomic relations. These seven cases are to be taken into account when computing the second composition, and the union of seven different compositions, as the composition table shows, always result in U.

This is explicitly shown for the first case:

$$\begin{aligned}
(m_1 \circ m_1) \circ m_1 &= r_1 \circ m_1 \\
&= \{m_1 m_2 m_4 m_5 m_8 m_{13} m_{14}\} \circ m_1 \\
&= m_1 \circ m_1 \cup m_2 \circ m_1 \cup \cdots \cup m_{14} \circ m_1 \\
&= r_1 \cup r_2 \cup r_4 \cup r_1^{\downarrow} \cup r_4^{\downarrow} \cup r_{13}^{\circlearrowleft} \cup r_{16}^{\circlearrowleft} \\
&= \{m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8 m_9 m_{10} m_{11} m_{12} m_{13} m_{14} m_{15} m_{16}\} \\
&= U \\
&= \{m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8 m_9 m_{10} m_{11} m_{12} m_{13} m_{14} m_{15} m_{16}\} \\
&= r_1 \cup r_5 \cup r_{13} \cup r_4 \cup r_{16} \cup r_2 \cup r_6 \\
&= m_1 \circ m_1 \cup m_1 \circ m_2 \cup \cdots \cup m_1 \circ m_{14}
\end{aligned}$$

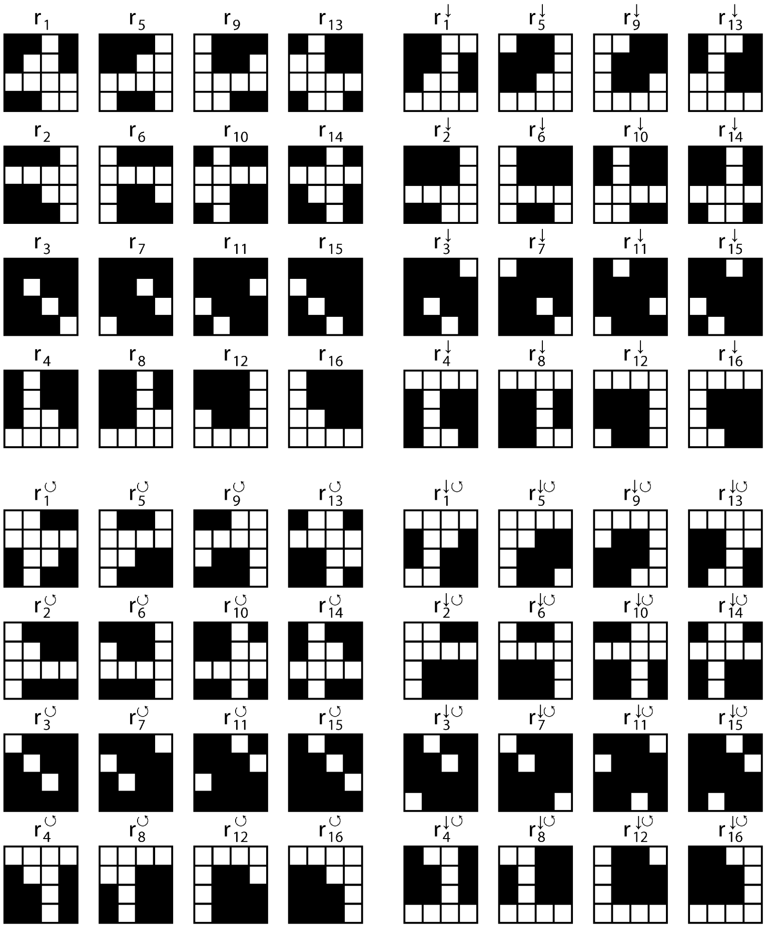

Fig. A1. The 64 non-atomic motion patterns found in the composition table.

$$= m_1 \circ \{m_1 m_2 m_4 m_5 m_8 m_{13} m_{14}\}$$
$$= m_1 \circ r_1$$
$$= m_1 \circ (m_1 \circ m_1).$$

(3) Taking the composition of one of the relations $m_i \in \mathcal{M}$ with the identity relation amounts to have exactly the same start positions and motion directions two times. The composition result is therefore the motion pattern resulting from the first start positions and first motion directions, which is equal to the first motion pattern m_i. That is, it holds that $\forall_{i \in 1..16} m_i \circ Id = m_i$ and consequently also for all non-atomic relations.

(4) As Fig. 6 shows, the converse for every atomic motion pattern is unique, and so is its converse. Fig. 6 explicitly defines $\forall_{i \in 1..16} \breve{\breve{m}}_i = m_i$.

(5)

$$(\{m_1\} \cup \{m_2\})\breve{} = (\{m_1 m_2\})\breve{}$$
$$= \{m_1 m_2\}\breve{}$$
$$= \{m_{11} m_{15}\}$$
$$= \{m_{11}\} \cup \{m_{15}\}$$
$$= \{\breve{m}_1\} \cup \{\breve{m}_2\}.$$

Since the converse is unique in each case (compare (4)), this can generally be put as follows:

$$\forall \mathrm{i},\mathrm{j} \in \{1,2,\ldots,16\} \quad \exists k,j : \breve{\mathrm{m}}_\mathrm{i} = \mathrm{m}_\mathrm{k} \wedge \breve{\mathrm{m}}_\mathrm{j} = \mathrm{m}_\mathrm{l},$$

$$\begin{aligned}(\{\mathrm{m_i}\} \cup \{\mathrm{m_j}\})\breve{} &= (\{\mathrm{m_i m_j}\})\breve{} \\ &= \{\mathrm{m_i m_j}\}\breve{} \\ &= \{\mathrm{m_k m_l}\} \\ &= \{\mathrm{m_k}\} \cup \{\mathrm{m_l}\} \\ &= \{\breve{\mathrm{m}}_\mathrm{i}\} \cup \{\breve{\mathrm{m}}_\mathrm{j}\}.\end{aligned}$$

(6)

$$\begin{aligned}(\mathrm{m_1} \circ \mathrm{m_1})\breve{} &= (\mathrm{r_1})\breve{} \\ &= (\{\mathrm{m_1 m_2 m_4 m_5 m_8 m_{13} m_{14}}\})\breve{} \\ &= \{\mathrm{m_{11} m_{15} m_7 m_{12} m_8 m_{10} m_{14}}\} \\ &= \mathrm{r}_5^{\downarrow\circlearrowleft} \\ &= \mathrm{m_{11}} \circ \mathrm{m_{11}} \\ &= \breve{\mathrm{m}}_1 \circ \breve{\mathrm{m}}_1,\end{aligned}$$

$$\begin{aligned}(\mathrm{m_2} \circ \mathrm{m_1})\breve{} &= (\mathrm{r_2})\breve{} \\ &= (\{\mathrm{m_1 m_2 m_3 m_9 m_{10} m_{13} m_{14} m_{15}}\})\breve{} \\ &= \{\mathrm{m_{11} m_{15} m_3 m_9 m_{13} m_{10} m_{14} m_2}\} \\ &= \mathrm{r}_6^{\downarrow\circlearrowleft} \\ &= \mathrm{m_{11}} \circ \mathrm{m_{15}} \\ &= \breve{\mathrm{m}}_1 \circ \breve{\mathrm{m}}_2,\end{aligned}$$

$$\begin{aligned}(\mathrm{m_3} \circ \mathrm{m_1})\breve{} &= (\mathrm{r_3})\breve{} \\ &= (\{\mathrm{m_1 m_2 m_3 m_4 m_5 m_7 m_8 m_9 m_{10} m_{12} m_{13} m_{14} m_{15}}\})\breve{} \\ &= \{\mathrm{m_2 m_3 m_4 m_5 m_7 m_8 m_9 m_{10} m_{11} m_{12} m_{13} m_{14} m_{15}}\} \\ &= \mathrm{r}_7^{\downarrow\circlearrowleft} \\ &= \mathrm{m_{11}} \circ \mathrm{m_3} \\ &= \breve{\mathrm{m}}_1 \circ \breve{\mathrm{m}}_3,\end{aligned}$$

$$\begin{aligned}(\mathrm{m_4} \circ \mathrm{m_1})\breve{} &= (\mathrm{r_4})\breve{} \\ &= (\{\mathrm{m_1 m_3 m_4 m_5 m_7 m_8 m_9 m_{12}}\})\breve{} \\ &= \{\mathrm{m_{11} m_3 m_7 m_{12} m_4 m_8 m_9 m_5}\} \\ &= \mathrm{r}_8^{\downarrow\circlearrowleft} \\ &= \mathrm{m_{11}} \circ \mathrm{m_7} \\ &= \breve{\mathrm{m}}_1 \circ \breve{\mathrm{m}}_4\end{aligned}$$

(7)

$$
\begin{aligned}
\breve{m}_1 \circ \overline{m_1 \circ m_1} \cap m_1 &= \breve{m}_1 \circ \overline{r_1} \cap m_1 \\
&= \breve{m}_1 \circ \overline{\{m_1 m_2 m_4 m_5 m_8 m_{13} m_{14}\}} \cap m_1 \\
&= \breve{m}_1 \circ \{m_3 m_6 m_7 m_9 m_{10} m_{11} m_{12} m_{15} m_{16}\} \cap m_1 \\
&= m_{11} \circ \{m_3 m_6 m_7 m_9 m_{10} m_{11} m_{12} m_{15} m_{16}\} \cap m_1 \\
&= (m_{11} \circ m_3 \cup m_{11} \circ m_6 \cup \cdots \cup m_{11} \circ m_{16}) \cap m_1 \\
&= (r_7 \cup r_{12} \cup r_8 \cup r_{13} \cup r_9 \cup r_5 \cup r_1 \cup r_6 \cup r_2)^{\downarrow\circlearrowleft} \cap m_1 \\
&= (r_7^{\downarrow\circlearrowleft} \cup r_{12}^{\downarrow\circlearrowleft} \cup r_8^{\downarrow\circlearrowleft} \cup r_{13}^{\downarrow\circlearrowleft} \cup r_9^{\downarrow\circlearrowleft} \cup r_5^{\downarrow\circlearrowleft} \cup r_1^{\downarrow\circlearrowleft} \cup r_6^{\downarrow\circlearrowleft} \cup r_2^{\downarrow\circlearrowleft}) \cap m_1 \\
&= \{m_2 m_3 m_4 m_5 m_6 m_7 m_8 m_9 m_{10} m_{11} m_{12} m_{13} m_{14} m_{15} m_{16}\} \cap m_1 \\
&= \emptyset,
\end{aligned}
$$

$$
\begin{aligned}
\breve{m}_2 \circ \overline{m_2 \circ m_1} \cap m_1 &= \breve{m}_2 \circ \overline{r_2} \cap m_1 \\
&= \breve{m}_2 \circ \overline{\{m_1 m_2 m_3 m_9 m_{10} m_{13} m_{14} m_{15}\}} \cap m_1 \\
&= \breve{m}_2 \circ \{m_4 m_5 m_6 m_7 m_8 m_{11} m_{12} m_{16}\} \cap m_1 \\
&= m_{15} \circ \{m_4 m_5 m_6 m_7 m_8 m_{11} m_{12} m_{16}\} \cap m_1 \\
&= (m_{15} \circ m_4 \cup m_{15} \circ m_5 \cup \cdots \cup m_{15} \circ m_{16}) \cap m_1 \\
&= (r_3 \cup r_{16} r_{12} \cup r_8 \cup r_4 \cup r_5 \cup r_1 \cup r_2)^{\circlearrowleft} \cap m_1 \\
&= (r_3^{\circlearrowleft} \cup r_{16}^{\circlearrowleft} \cup r_{12}^{\circlearrowleft} \cup r_8^{\circlearrowleft} \cup r_4^{\circlearrowleft} \cup r_5^{\circlearrowleft} \cup r_1^{\circlearrowleft} \cup r_2^{\circlearrowleft}) \cap m_1 \\
&= \{m_2 m_3 m_4 m_5 m_6 m_7 m_8 m_9 m_{10} m_{11} m_{12} m_{13} m_{14} m_{15} m_{16}\} \cap m_1 \\
&= \emptyset,
\end{aligned}
$$

$$
\begin{aligned}
\breve{m}_3 \circ \overline{m_3 \circ m_1} \cap m_1 &= \breve{m}_3 \circ \overline{r_3} \cap m_1 \\
&= \breve{m}_3 \circ \overline{\{m_1 m_2 m_3 m_4 m_5 m_7 m_8 m_9 m_{10} m_{12} m_{13} m_{14} m_{15}\}} \cap m_1 \\
&= \breve{m}_3 \circ \{m_6 m_{11} m_{16}\} \cap m_1 \\
&= m_3 \circ \{m_6 m_{11} m_{16}\} \cap m_1 \\
&= (m_3 \circ m_6 \cup m_3 \circ m_{11} \cup m_3 \circ m_{16}) \cap m_1 \\
&= (r_6 \cup r_9 \cup r_{16}) \cap m_1 \\
&= \{m_2 m_3 m_4 m_6 m_7 m_8 m_{10} m_{11} m_{12} m_{14} m_{15} m_{16}\} \cap m_1 \\
&= \emptyset,
\end{aligned}
$$

$$
\begin{aligned}
\breve{m}_4 \circ \overline{m_4 \circ m_1} \cap m_1 &= \breve{m}_4 \circ \overline{r_4} \cap m_1 \\
&= \breve{m}_4 \circ \overline{\{m_1 m_3 m_4 m_5 m_7 m_8 m_9 m_{12}\}} \cap m_1 \\
&= \breve{m}_4 \circ \{m_2 m_6 m_{10} m_{11} m_{13} m_{14} m_{15} m_{16}\} \cap m_1 \\
&= m_7 \circ \{m_2 m_6 m_{10} m_{11} m_{13} m_{14} m_{15} m_{16}\} \cap m_1 \\
&= (m_7 \circ m_2 \cup m_7 \circ m_6 \cup \cdots \cup m_7 \circ m_{16}) \cap m_1 \\
&= (r_7 \cup r_6 \cup r_5 \cup r_9 \cup r_4 \cup r_8 \cup r_{12} \cup r_{16})^{\downarrow} \cap m_1 \\
&= (r_7^{\downarrow} \cup r_6^{\downarrow} \cup r_5^{\downarrow} \cup r_9^{\downarrow} \cup r_4^{\downarrow} \cup r_8^{\downarrow} r_{12}^{\downarrow} \cup r_{16}^{\downarrow}) \cap m_1 \\
&= \{m_2 m_3 m_4 m_5 m_6 m_7 m_8 m_9 m_{10} m_{11} m_{12} m_{13} m_{14} m_{15} m_{16}\} \cap m_1 \\
&= \emptyset.
\end{aligned}
$$

The validity of the axioms prove the proposed relational system with the relations in $\mathscr{M}$ to be a relation algebra.

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